Use of Turbulent Kinetic Energy in Free Mixing Studies

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The concept that the turbulent kinetic energy equation can be used to determine the shear in a turbulent flowfield through the use of a suitable relation between turbulent shear and turbulent kinetic energy has proved successful in the analysis of turbulent boundary-layer flows. In this paper, the application of a similar approach to the problem of turbulent free mixing of constant-density streams is described. By correlating measurements of turbulent shear and turbulent kinetic energy in a number of constant density free mixing flows, a linear relation between turbulent shear and turbulent kinetic energy is shown to exist. The combination of this relationship and a new rapid technique for the simultaneous solution of an arbitrary number of parabolic partial differential equations allows detailed calculation to be carried out for two-steam mixing systems of interest, one a plane mixing region and the other axisymmetric. Both mixing regions are constant density. Generally satisfactory agreement is achieved for both velocity and turbulent shear distribution. More important than the level of agreement reached, however, is the fact that the method used is more perceptive than previous phenomenological approaches and, thus, offers the promise of eventually leading to greater understanding of turbulent shear flow.

Nomenclature

a = coefficient of the general parabolic equation
a_1 = constant in relation between turbulent shear and turbulent kinetic energy
a_2 = constant in turbulent kinetic energy dissipation relation
b,c,d = coefficients of the general parabolic equation
D_k = turbulent kinetic energy dissipation
J_k = turbulent kinetic energy flux
k = kinetic energy of turbulence
l = Prandtl mixing length
l_k = mixing length for turbulent kinetic energy
$\dot{M}$ = entrainment flow rate
p = pressure
U = mean stream velocity
u,v = time-averaged velocity components
u',v',w' = fluctuating turbulent velocity components
x,y = independent coordinates
α = parameter; $\alpha = 0$ for plane flow, $\alpha = 1$ for axisymmetric flow
ϵ = eddy viscosity
ρ = density
σ_k = turbulent kinetic energy quantity analogous to Prandtl number for total mean-flow energy
τ = turbulent shear stress
φ = general dependent variable
ψ = stream function
ω = dimensionless stream function

Subscripts

E = external boundary
I = internal boundary
1 = outer stream
2 = inner stream

Other symbols

$\langle\ \rangle$ = indicates time average of quantity

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I. Introduction

FREE turbulent mixing is a process of fundamental importance to the description of phenomena such as occur in the flowfield downstream of an aerodynamic body and in the region of interaction between fuel and oxidizer in a combustion system. A number of approaches to the analysis of this process have been explored, with varying degrees of success. In common with the analysis of other turbulent shear flows, analytical models of free turbulent mixing have often used the concept of the eddy viscosity, introduced by Boussinesq, which allows the governing equations to be reduced to a form similar to the governing equations in laminar flow. In this formulation, the turbulent shear stress, which is defined by the equation

$$\tau = -\langle(\rho v)'u'\rangle \tag{1}$$

is related to the local mean flow velocity gradient, using the eddy viscosity ϵ. This results in the expression

$$\tau = \epsilon(\partial u/\partial y) \tag{2}$$

in which the eddy viscosity ϵ replaces for a turbulent flow the molecular viscosity in a laminar flow. In such an approach, the theoretical problem becomes one of predicting the behavior of the eddy viscosity, which, unlike the physical viscosity, is a function of the particular flowfield.

The most commonly used models for the eddy viscosity in turbulent flow analyses are the several variations of the mixing length model proposed originally by Prandtl. In the Prandtl model, the eddy viscosity is itself related to the mean flow velocity gradient through the expression

$$\epsilon = \rho l^2|\partial u/\partial y| \tag{3}$$

in which l is an experimentally determined parameter, called the mixing length. Thus in this formulation, the turbulent shear is given by the relation

$$\tau = \rho l^2|\partial u/\partial y|\ \partial u/\partial y \tag{4}$$

The relationship expressed by Eq. (4) has been reasonably successful in the prediction of mean flow quantities in some flowfields, but it has not proved feasible to extend it to cover a wide range of flow conditions using a single empirically determined mixing length. Such a result is not entirely unexpected, for it is known that the Prandtl model is physically unrealistic because it relates the eddy viscosity only to the local gradient of mean flow velocity. Thus, there is no history effect incorporated in this model; in it the eddy viscosity at a point, and thus the turbulent shear in the region around the point, is determined only by local conditions without regard to the prior development of the flow.

This observation has led to the exploration of more physically perceptive models for the turbulent shear stress. Such a model was suggested by Nevzgljadov,[1] and discussed by Dryden.[2] In this model, the local turbulent shear is linearly related to the local kinetic energy of turbulence through the relation

$$\tau = a_1 \rho k \tag{5}$$

in which k represents the turbulent kinetic energy, defined by

$$k = \tfrac{1}{2}[\langle u'^2 \rangle + \langle v'^2 \rangle + \langle w'^2 \rangle] \tag{6}$$

The constant a_1 is determined empirically. With Eq. (5), the turbulent kinetic energy equation may be used as an additional equation to determine the development of the turbulent shear field. Thus this relation allows the turbulent shear at a station in the flowfield to be affected by the development of the flowfield to that station, incorporating therefore the history of the flow. Such a linear relation between local turbulent shear and local turbulent kinetic energy was used by Bradshaw, Ferriss, and Atwell[3] in the study of wall boundary layers, with good results, and it seemed of interest to attempt to apply the approach to the problem of free turbulent mixing.

The application of this approach to the problem at hand rests, however, on the hypothesis that a linear relation between turbulent kinetic energy and turbulent shear exists in free mixing, so that the constant a_1 in Eq. (5) can be evaluated. To explore this hypothesis, experimental measurements of turbulent shear and turbulent kinetic energy were obtained from the literature for a wide variety of turbulent free mixing flows. The experiments of Lee[4] were concerned with plane mixing between two airstreams, at velocity ratios of 1.0 and 2.857; those of Zawacki and Weinstein[5] were concerned with the axisymmetric mixing between coaxial airstreams, with outer stream to inner stream velocity ratios ranging from 1.0 to 39.5, whereas the experiments of Heskestad[6,7] were concerned with both a plane jet and a radial jet issuing into still air. The measured turbulent shear for all of these experiments is plotted in nondimensional form in Fig. 1 vs the nondimensional turbulent kinetic energy. From Fig. 1, it is clear that a linear relationship does exist for free turbulent mixing, with the value of the constant a_1 in Eq. (5) taken as 0.3.‡ This value of a_1 is identical to that obtained by Bradshaw, Ferriss, and Atwell[3] for boundary-layer flows. It should be noted that the definition of k, Eq. (6), used in this work differs by a factor of $\frac{1}{2}$ from that used in Ref. 3.

In the work of Bradshaw, Ferriss, and Atwell, the turbulent kinetic energy equation is recast into an equation for the turbulent shear stress through the use of Eq. (5). An important feature of their work is their observation that, when the kinetic energy equation is rewritten in this manner, the system of equations (momentum and kinetic energy) applicable to the incompressible wall boundary-layer problem becomes hyperbolic. The solution to the system may thus be approached using the method of characteristics, with all of the attendant advantages of computational efficiency. However, such a hyperbolic system has the disadvantage that when other equations, such as the mean flow energy equation and the species equation, are incorporated into the system,

‡ The original version of this paper reflected an inadvertent error in the determination of the parameter a_1, which was reported as 0.2. The calulations reported here have been revised using the correct value, $a_1 = 0.3$; the earlier results are also included for purposes of comparison. The remarks made in Sec. IV refer to the revised calculations.

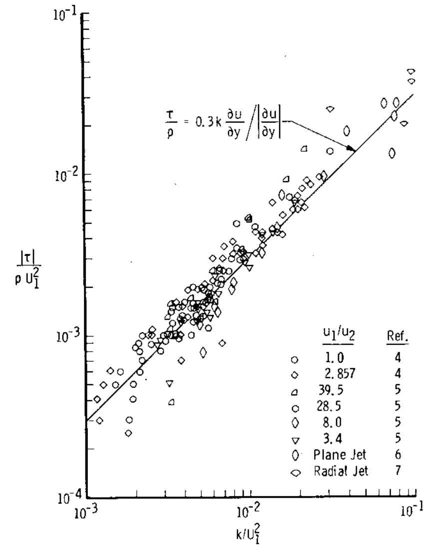

Fig. 1 Correlation between local turbulent shear stress and local turbulent kinetic energy.

as is appropriate in the study of more complex problems, the method becomes unwieldy. If the momentum and kinetic energy equations are written in terms of the eddy viscosity ϵ treated as a parameter which is defined by Eq. (2), they are parabolic. The ultimate goal of the present study is to analyze free mixing systems in which mixing of dissimilar gases occurs. Treatment of such problems will require the solution of the mean flow energy and species equations, both of which are expressible in parabolic form, in addition to the momentum and kinetic energy equations. To facilitate the eventual extension of the approach to more complex free mixing systems, the momentum and turbulent kinetic energy equations have been maintained in parabolic form, with the relation between turbulent shear and turbulent kinetic energy serving as an auxiliary equation.

A further reason for maintaining the parabolic form of the equations of motion is that it then becomes possible to use the computational technique developed by Patankar[8] to obtain the solution of the system. In this technique, numerical solutions to an arbitrary number of simultaneous parabolic differential equations are obtained through use of a variable-grid finite difference method, in which the lateral grid size and the forward step size are both controlled by the computed rate of growth of the mixing layer. Because of this variable grid, the time required to solve the equation system desired is considerably reduced from that necessary with a fixed grid size. The resulting computational efficiency is of considerable importance in an analysis such as the present one in which, because of the incorporation of the turbulent kinetic energy equation, even the simplest problem requires the simultaneous solution of two equations.

The combination of the computation technique developed by Patankar and the relation represented by Eq. (5) makes it possible to analytically approach the problem of free turbulent mixing from somewhat more fundamental grounds than is possible using phenomenological models such as described by Eq. (4). To begin a general study of turbulent free mixing using the linear relation between turbulent shear and turbulent kinetic energy, the analysis of two simple mixing systems was undertaken. One of the systems involved plane free mixing, and the other axisymmetric; both were constant density.

II. Governing Equations

The governing equations for free mixing are partial differential equations of the boundary layer type. The detailed development of these equations has been discussed by Townsend[10] and Hinze.[11] For two-dimensional plane or axisymmetric flow, the equations of motion are as follows.

Continuity

$$\partial \rho u y^\alpha / \partial x + \partial \rho v y^\alpha / \partial y = 0 \tag{7}$$

where u and v are the time average velocity in the x and y directions of the flowfield, respectively, and ρ is the density which is here taken to be constant. The parameter α equals 0 for plane flow and 1 for axisymmetric flow.

Momentum

In parabolic form, the momentum equation is written

$$\rho u(\partial u/\partial x) + \rho v(\partial u/\partial y) = y^{-\alpha}(\partial/\partial y)\,[y^\alpha \epsilon(\partial u/\partial y)] - \partial p/\partial x \tag{8}$$

where p is the static pressure, which can be considered to be constant for free mixing, and ϵ is the eddy viscosity, defined by Eq. (2). The shear stress τ which appears in Eq. (2) can be evaluated through Eq. (5). However, it has been observed that the kinetic energy of turbulence is always positive whereas the shear stress may be either positive or negative. Since the local shear stress has the same sign as the local velocity gradient, Eq. (5) has been modified by multiplying it by the ratio of the local velocity gradient to the absolute value of that gradient

$$\tau/\rho = a_1 k(\partial u/\partial y)/|\partial u/\partial y| \tag{9}$$

There remains an additional difficulty in the use of Eq. (9), that arises from the observation that the local shear stress approaches zero simultaneously with the local velocity gradient while the local kinetic energy of turbulence may still remain nonzero. It was shown in Fig. 1 that Eq. (9) correlates well with experimental measurements of turbulent shear vs turbulent kinetic energy; Fig. 1, however, does not include the measured data in the region of zero shear stress. The situation in which shear can approach zero while the local kinetic energy of turbulence remains nonzero generally exists in the vicinity of a relative maximum or minimum in the velocity profile. For example, the initial boundary layer can cause a minimum velocity to occur in the profile of the far downstream region in the case of free mixing between equal velocity streams, whereas a decaying jet always has a relative maximum in the velocity profile. In order to make the modified relation, Eq. (9), applicable in these particular regions, a restriction is imposed so that

$$(\tau/\rho)_{\partial u/\partial y \simeq 0} = (\tau/\rho)|\partial u/\partial y|/|\partial u/\partial y|_{\max} \tag{10}$$

where $|\partial u/\partial y|_{\max}$ is the maximum absolute value of the velocity gradient at each cross section of the mixing region.§

Turbulent Kinetic Energy

In parabolic form, the turbulent kinetic energy equation is written[9]

$$\rho u(\partial k/\partial x) + \rho v(\partial k/\partial y) = y^{-\alpha}(\partial/\partial y)\left[y^\alpha\,(\epsilon/\sigma_\kappa)\,\frac{\partial k}{\partial y}\right] + \epsilon(\partial u/\partial y)^2 - D_\kappa \tag{11}$$

§ In the interval since the original version of this paper was presented, radial distributions of a_1 have been obtained from Ref. 12 for axisymmetric flow. It has been found that these distributions can be approximated by applying a correction similar to that embodied in Eq. (10) from the flow centerline to the radius of maximum shear. This modification has been made to all of the axisymmetric calculations presented herein.

The quantity ϵ/σ_k is the exchange coefficient for the turbulent kinetic energy flux in the y direction, defined by

$$\epsilon/\sigma_k = -J_k/(\partial k/\partial y) = -\langle(\rho v)'k\rangle/(\partial k/\partial y) \tag{12}$$

The physical significance of σ_k for turbulent kinetic energy is equivalent to that of the Prandtl number for total energy.

The term D_k represents the dissipation of the turbulent energy, which has been discussed by Townsend[10] and Hinze[11] for homogeneous, isotropic turbulence. For nonisotropic turbulence, however, the dissipation term has yet to be studied. An expression for the dissipation term discussed by Patankar and Spalding[9] and used by Bradshaw et al.,[3] takes the form

$$D_k = a_2 \rho k^{3/2}/l_k \tag{13}$$

where a_2 is an empirical constant and l_k is analogous to the Prandtl mixing length. In this study, l_k is assumed to be equal to the width of the mixing region. It is necessary to point out that the constant a_2 is not known and that further experimental work is needed to explore the proper form of the dissipation function.

The problem of free turbulent mixing between incompressible airstreams may now be considered as a standard mathematical problem involving five unknowns, u, v, k, τ, and ϵ, with five simultaneous equations, Eqs. (2, 7–9, and 11).

III. Method of Solution

The method developed by Patankar[8] for numerical solution of simultaneous parabolic differential equations appears to be the most advanced scheme in current use. Patankar used the von Mises transformation to establish a streamline coordinate system by defining the stream function ψ such that

$$\rho u y^\alpha = \partial\psi/\partial y \tag{14}$$

and

$$\rho v y^\alpha = -\partial\psi/\partial x \tag{15}$$

thus satisfying the continuity equation. By introducing a dimensionless stream function

$$\omega = (\psi - \psi_I)/(\psi_E - \psi_I) \tag{16}$$

with the subscripts I and E designating the internal and external boundaries, respectively, the equations of motion take on the general form

$$\partial\varphi/\partial x + (a + b\omega)\partial\varphi/\partial\omega = (\partial/\partial\omega)(c\partial\varphi/\partial\omega) + d \tag{17}$$

where φ, in this paper, may be either the velocity component u or the turbulent kinetic energy, k. The constants a, b, and c take the following form:

$$a = y_I{}^\alpha \dot{M}_I/(\psi_E - \psi_I) \tag{18}$$

$$b = (y_E{}^\alpha \dot{M}_E - y_I{}^\alpha \dot{M}_I)/(\psi_E - \psi_I) \tag{19}$$

$$c = y^{2\alpha}\rho u\epsilon/[(\psi_E - \psi_I)^2\sigma] \tag{20}$$

where σ may be 1 or σ_k as φ represents u or k, respectively. The terms $y_I{}^\alpha\dot{M}_I$ and $y_E{}^\alpha\dot{M}_E$ are the entrainment flow rates at the I or E boundaries, respectively, and are defined by the equation

$$y_I{}^\alpha\dot{M}_I + \omega_B(y_E{}^\alpha\dot{M}_E - y_I{}^\alpha\dot{M}_I) = \left[\frac{(\partial/\partial\omega)\,[y^{2\alpha}\rho u\epsilon/(\psi_E - \psi_I)]\,\partial u/\partial\omega}{\partial u/\partial\omega}\right]_B - \frac{\psi_E - \psi_I}{(\partial u/\partial\omega)_B}\left(\frac{du_B}{dx}\right) \tag{21}$$

Equation (21) is used to control the variation of the finite difference grid, by controlling the rate of growth of the mass flow in the mixing region. It is to be evaluated at some value of ω_B, arbitrarily chosen, at which it is desired that the velocity in the mixing region u_B be a given fraction of the external stream velocity. For example, one might choose that, at $\omega_B = 0.95$, $u_B = 0.99\ U_1$ where U_1 is the external stream velocity. The term du_B/dx is expressed in finite difference form by

$$du_B/dx \simeq (u_{BD} - u_B)/\Delta x \tag{22}$$

where $u_{BD} - u_B$ is the difference in the velocity desired at the downstream step u_{BD} and the velocity actually computed u_B and Δx is the step length. Thus Eq. (22) serves to correct the approximation made for the entrainment at the upstream step. The effect of this correction is to alter the axial step size so that a poor approximation for the entrainment results in a very small axial step length between the step for which the entrainment was calculated and the succeeding step. Equation (21) is evaluated at values of ω_B near both edges of the mixing region for plane flow; for axisymmetric flow $\dot{M}_I = 0$.

Because Eq. (21) is arbitrary and serves merely to control the grid width (and thus the computational efficiency), it would be expected that the results of the calculation would be insensitive to variation of the choice of the u_{BD} desired at a given ω_B; that is, there should be no change in the results (except for computational time) if u_{BD} is changed from 0.99 U_1 to 0.999 U_1. This was indeed observed to be the case.

The expressions for d in the general parabolic equation, Eq. (17), for the corresponding φ values are

$$\begin{aligned} d &= -(1/\rho u)\partial p/\partial x, \text{ for } \varphi = u \\ d &= [y^{2\alpha}\rho u\epsilon/(\psi_E - \psi_I)^2](\partial u/\partial\omega)^2 - D_k/\rho u, \text{ for } \varphi = k \end{aligned} \tag{23}$$

A program for the numerical solution of an arbitrary number of simultaneous equations of the form of the general parabolic equation, Eq. (17), was written in FORTRAN IV language by Patankar. Readers who are interested in the details of the programing may consult Ref. 8.

IV. Comparison with Experimental Results

In order to explore the validity of using the turbulent energy equation, it was considered necessary that the selected experiments meet the requirements that 1) both the region dominated by the initial boundary layers and the region satisfying profile similarity are included and 2) the region in which the local shear stress approaches zero, which may or may not be accompanied by zero turbulent kinetic energy, is included.

Because of the necessity of obtaining accurate initial conditions for both velocity and turbulent kinetic energy profiles, only a limited number of experimental results are available to be used for comparison. The experiments of Lee[4] and Zawacki and Weinstein[5] satisfied the requirements and were compared with the analytical solutions.

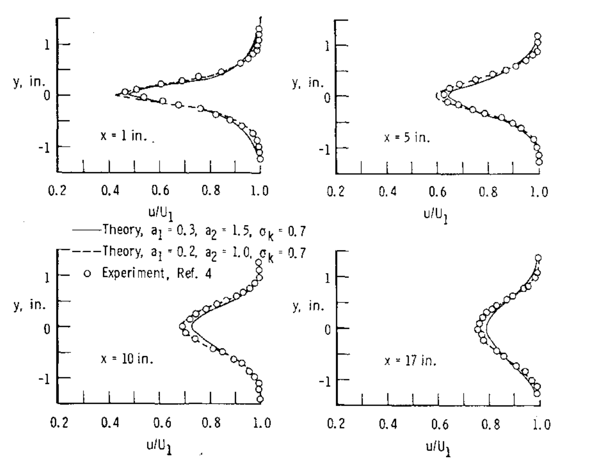

Fig. 2 Average velocity distributions, $U_1/U_2 = 1.0$.

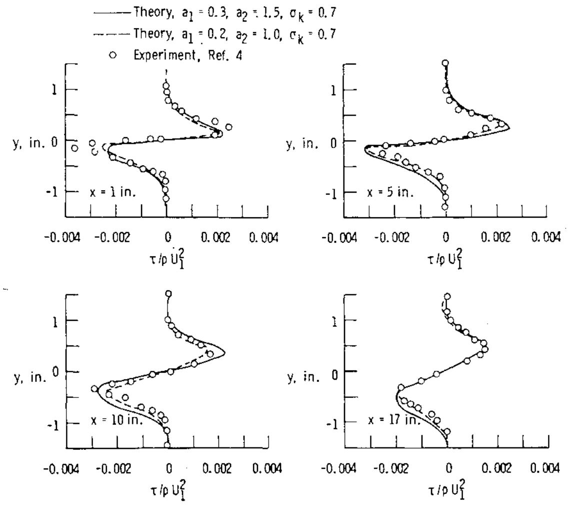

Fig. 3 Shear stress distributions, $U_1/U_2 = 1.0$.

Plane Flow

The experiments of Lee were concerned with free turbulent mixing between incompressible air streams with negligible pressure gradients, for freestream velocity ratios of 1.0 and 2.857.

Velocity ratio 1.0

The case of free mixing between equal velocity streams with initial boundary layers provides a good opportunity for investigating the region in the vicinity of the minimum velocity where the turbulent shear stress approaches zero while the turbulent kinetic energy remains nonzero. The restriction imposed by Eq. (10) was used in this region, which encompassed approximately 10% of the total mass flow in the mixing region, whereas the relation given by Eq. (9) was used for all other locations. A comparison of analytical and experimental results is shown in Fig. 2 for average velocity distributions; the agreement is surprisingly good. The velocity distributions in the downstream region were found to be strongly influenced by the initial conditions. The analytical result shown in Fig. 2 was obtained by using the measured initial condition to begin the marching solution of the parabolic equations. A comparison of the analytical and experimental results for the shear stress distributions at several axial locations is shown in Fig. 3; again the agreement as to both profile shape and magnitude of the shear stresses is quite good. The numerical constants used in the analysis are as follows:

$$a_1 = 0.3,\ a_2 = 1.5,\ \sigma_k = 0.7$$

The magnitude of a_1 was obtained from the experimental data shown in Fig. 1. However, because of the lack of sufficient experimental information, the numerical values of σ_k and a_2 were determined by a trial-and-error process.¶

Velocity ratio 2.857

The same numerical constants were used for the velocity ratio 2.857 case. Comparison of analytical and experimental profiles for average velocity and turbulent shear stress are shown in Figs. 4 and 5, respectively. The agreement for

¶ In the earlier version of this paper, with $a_1 = 0.2$, $a_2 = 1.0$ was found to give the best agreement. In both sets of calculations, $\sigma_k = 0.7$ was used.

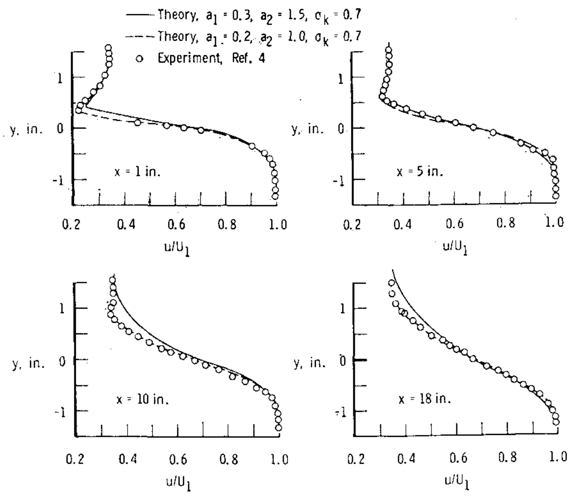

Fig. 4 Average velocity distributions, U_1/U_2 = 2.857.

average velocity is again good. The turbulent shear stress, however, agrees only in profile shape. In the early part of the flow, the analytical solution appears to overpredict the rate of increase of the turbulent kinetic energy; in the latter part, the dissipation of turbulent kinetic energy appears to be overpredicted. This comparison seems to suggest that the functional form of the dissipation expression, Eq. (13), is inadequate for this flow.

Axisymmetric Flow

The experiments of Zawacki and Weinstein[5] were concerned with free turbulent mixing between constant density coaxial airstreams with a negligible axial pressure gradient for ratios of freestream to jet velocity from 1.0 to 39.5. The outer stream velocity was fixed at 48 fps, while the inner stream velocity was varied to provide the desired velocity ratios; in all cases, the outer stream velocity was the greater of the two.

Zawacki and Weinstein encountered some difficulties in obtaining reliable measurements of Reynolds stresses in the near field region which created some problems in selecting initial boundary conditions for both average velocity and turbulent shear stress. By using the first reported profiles as the initial conditions at approximately one and one-half nozzle diameters downstream, calculations of the axisymmetric mixing process were made for the cases of velocity ratios of 8.0 and 39.5. The numerical values for a_1, a_2, and σ_k established in the plane mixing cases were again used for the axisymmetric case.

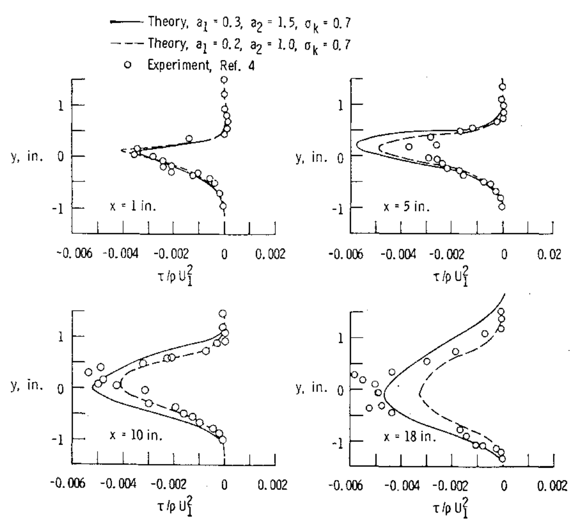

Fig. 5 Shear stress distributions, U_1/U_2 = 2.857.

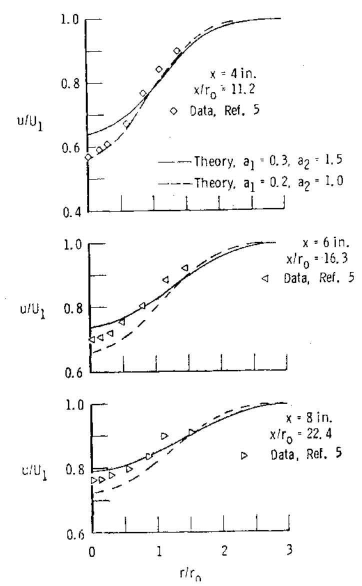

Fig. 6 Average velocity distributions, axisymmetric case, U_1/U_2 = 8.0.

Velocity Ratio 8.0

The velocity and shear stress distributions for the velocity ratio 8.0 case are shown in Figs. 6 and 7, respectively. It can be seen that the analytically determined average velocities agree reasonably well with the experimental results. However, the predicted turbulent shear appears to be two to three times larger than the measured values. In an attempt to accelerate the dissipation of kinetic energy, the constant a_2 that modifies the dissipation was increased by 50%. This reduced the predicted shear stress by 20%, which was still not sufficient to make the results comparable with the experimental data. Moreover, the increase in the value of a_2 changed the predicted velocity distribution, forcing it to deviate further from experiment.

Velocity Ratio 39.5

Figures 8 and 9 show the comparison between analytical and experimental results for velocity and turbulent shear for the velocity ratio 39.5 case. The constants were again those used in previous cases. The agreement for mean velocity is reasonably good; for shear velocity the agreement can be considered only qualitative. However, in defense of the method, it should be pointed out that, for the velocity ratio

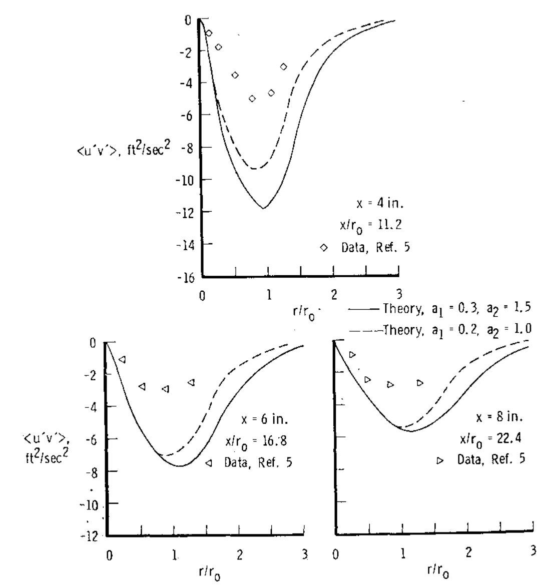

Fig. 7 Shear stress distributions, axisymmetric case, U_1/U_2 = 8.0.

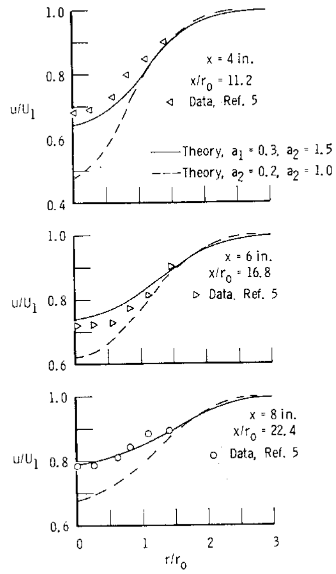

Fig. 8 Average velocity distributions, axisymmetric case, $U_1/U_2 = 39.5$.

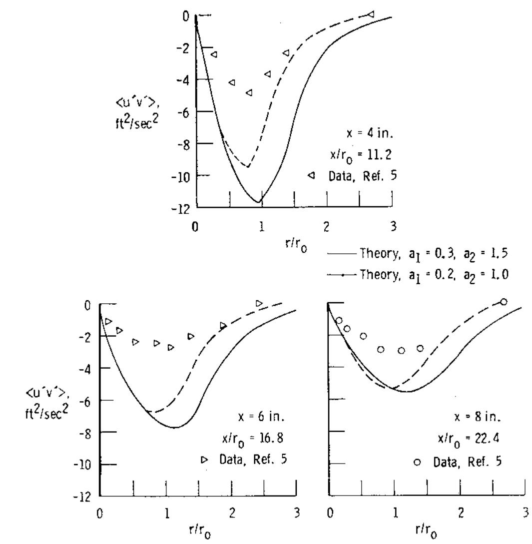

Fig. 9 Shear stress distributions, axisymmetric case, $U_1/U_2 = 39.5$.

39.5 case, the experimental inner jet velocity was on the order of 1 fps. In such a flow, which might be expected to behave like a wake with small base bleed, the possibility of there being significant departure from the assumption of zero axial pressure gradient made in the calculation should not be overlooked.**

V. Effect of Parameter Variation

As part of the study reported in this paper, numerous calculations were made with various values of the parameters, σ_k, a_1, and a_2. In this section, a brief qualitative summary of the behavior will be undertaken.

Variation with σ_k

The turbulent kinetic energy Prandtl number σ_k was varied over a range from 0.5 to 0.7, without change in the other parameters, in calculation of the two cases comprising the experimental work of Lee.[4] No significant change in the results was observed, indicating that the flowfields considered are not strongly affected by the kinetic energy Prandtl number, at least over a reasonable range of variation.

Variation with a_2

As the parameter a_2 directly affects the magnitude of the dissipation term, it might be expected that variation in a_2 would have a fairly large influence on the mixing process. In calculations as described above for σ_k, the value of a_2 was varied from 1.0 to 2.0. As a_2 was increased through this range, a decrease in the rate of mixing was observed. Quantitatively, for the velocity ratio 1.0 case of Lee,[4] the minimum velocity at the station 17 in. downstream from the beginning of the mixing region increased about 5% as a_2 was increased from 1.0 to 2.0.

Variation with a_1

Since the value of the parameter a_1 was obtained directly from experimental data, it was not changed in the course of any of the calculations described in the main body of this paper. Several calculations were, however, performed specifically to observe the effect of variation in a_1. For these calculations, a_1 was varied from 0.18 to 0.22. It was observed that increasing a_1 had qualitatively the effect of increasing the rate of mixing. Again basing a quantitative evaluation on the change in the calculated minimum velocity at $x = 17$ in. for the velocity ratio 1.0 case, a 10% change in a_1 about a value of 0.2 caused an approximately 3% change in the minimum velocity.

** In a recent calculation, the experimental velocity profiles presented in Ref. 5 were used in the data reduction procedure used in Ref. 13 to determine the turbulent shear stress through integration of the mean flow momentum equation. The results indicated a shear stress level approximately twice that reported in Ref. 5, which makes the comparison between theory and experiment considerably more favorable to the theory.

VI. Conclusions and Recommendations

A study of the use of the turbulent kinetic energy equation in the analysis of free mixing problems was conducted. The analytical results obtained were compared with the available experimental data for both plane and axisymmetric flows, with generally satisfactory agreement for both velocity and shear stress distributions. The concept that the shear stress in turbulent flow can be considered as an additional dependent variable to be solved for simultaneously with other related flow parameters seems justified. However, in order to improve the current level of understanding of turbulence characteristics in free mixing, further investigation is necessary in several areas.

Experimental Work

Turbulence correlation measurements are very much needed to establish a realistic model for dissipation of turbulence, as are Reynolds stress measurements, especially in the region in which the shear stress approaches zero while the turbulent kinetic energy remains nonzero.

Analytical Work

The possibility exists that the constants involved in the turbulent energy equation may be applicable to more complicated mixing problems than the incompressible free mixing considered here. Analytical studies that use the turbulent energy approach for compressible air-air and hydrogen-air mixing are being developed. A systematic investigation of mixing problems may conceivably lead to orderly functional relationships for the parameters involved. Consequently, the uncertainties in the exchange coefficients in turbulent flow problems may eventually be eliminated.

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Transpiration and Film Cooling Effects for a Slender Cone in Hypersonic Flow

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Tests were conducted in the LTV Aerospace Corporation (Vought Aeronautics Division) Hypervelocity Wind Tunnel to determine transpiration and film cooling effects on the aerodynamic characteristics of a slender cone in hypersonic flow. Film cooling of the model was obtained by injecting methane out of and near the slightly truncated nose. Transpiration cooling was provided by injecting ethylene through the remaining surface of the model. The resulting effects on heat transfer, skin friction, and pressure distribtions, and force and moment data are analyzed and discussed. The flow conditions were limited to nominal Mach numbers of 12 and 17, and respective Reynolds numbers per ft of 6 × 10⁶ and 10⁶. It was found that the heat transfer, skin friction, axial force coefficient, and normal force coefficient slope decreased with increased mass injection for both film and transpiration cooling. The effectiveness of film cooling, however, decreased with distance from the injection region, and diminished with increased angle of attack. It was observed from test results and photographs that boundary-layer transition was induced by light mass injection at the higher Reynolds number.

Nomenclature

A = area
C = Chapman-Rubesin coefficient, $\rho\mu/(\rho\mu)_{\rm ref}$
C_A = axial force coefficient with base pressure assumed to be zero
C_A' = axial force coefficient with base pressure equal to the free-stream value
C_D = drag coefficient
C_f = skin-friction coefficient
C_i = injection parameter, $\dot{m}/\rho_\infty V_\infty A_b$
C_M = moment coefficient
C_N = normal force coefficient
C_P = pressure coefficient
c_p = specific heat at constant pressure
d = diameter
k = thermal conductivity
L = model length
$\dot{m}$ = injected mass flow rate (total)
M = Mach number
P = static pressure
Pr = Prandtl number, $c_p\mu/k$
$\dot{q}$ = heat-transfer rate
r = radius
Re_x = Reynolds number, $\rho_c V_c x/\mu_c$
St = Stanton number, $\dot{q}/\rho c_p V(T_r - T_w)$
t = time
T = temperature
V = inviscid stream velocity
W = vehicle weight
x = station on cone (distance from apex)
x' = axial distance from aftmost point of mass injection
α = angle of attack
β = ballistic coefficient $W/C_D A_b$
γ = entry angle
θ_c = cone half angle
μ = viscosity
ρ = density

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